Optimal Continuous Torque Attitude Maneuvers

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A nonsingular formulation of the necessary conditions for optimal large-angle rotational maneuvers is presented. By optimal we mean that some integral measure of the system performance is minimized. The formulation is for the case of a rigid asymmetric vehicle having a control system capable of generating a continuous torque history; it is valid for an arbitrary specification of terminal orientation and angular velocity states. Analytical solutions of the two-point boundary-value problem are extracted for single-axis maneuvers, and a relaxation process is proposed for reliable numerical solutions for arbitrary boundary conditions. The validity of the necessary condition formulation is supported by analytical solutions for special cases; the relaxation process for general boundary conditions is illustrated by numerical examples.

I. Introduction

OPTIMAL, continuous torque, large-angle reorientations of asymmetric, generally tumbling spacecraft have not received a formal treatment in the literature. There are a number of feasible large-angle maneuver strategies established,[1-6] and small-angle attitude control falls within the class of problems to which the now-classical optimal linear regulator theory[7] applies. While the nonlinear optimal rotational maneuver problem has not been treated per se, it is true that the problem's mathematical structure lies within the class of problems addressed by several rather general formulations[7-12] of optimal control theory. Unfortunately, formulation and solution of optimal control problems are two distinct plateaus: the first plateau being several orders of magnitude more easily reached for most problems.

It is a well-known truth that the application of a variational extremum principle (e.g., Pontryagin's Principle) to a dynamical system leads to a system of differential equations whose order is double that of the original system. In addition, the boundary conditions are usually split, with typically half of the boundary conditions being specified at the initial time, and the rest being specified at the final time. For most cases the differential equations are nonlinear, and the resulting two-point boundary-value problem (TPBVP) resists solution except by iteration. As a consequence, many trial solutions may be required to determine one or more boundary condition "ballparks" in which to initiate a successive approximation algorithm.

The expense, frustration, and often, failure of various strategies for solution of this type of TPBVP have motivated development of numerous approximate optimization algorithms (e.g., quasilinearization methods, dynamic programming/invariant imbedding,[13-15] parameterization/parameter optimization,[16-18] and function space gradient methods).[19] All of these methods can be viewed as means for establishing approximate optimal trajectories; satisfactory convergence can usually be achieved, but depends greatly upon the intuitive and empirical skills of the user. In lieu of these various methods, we will discuss an iterative relaxation process for the solution of the problem which makes efficient use of the information obtained from the closed-form solution for the single axis special case. The solutions so determined rigorously satisfy Pontryagin's necessary conditions everywhere to an arbitrary (machine dependent) precision.

II. Formulation of Nonsingular Optimal Rotational Maneuver Necessary Conditions

A. Orientation and Rotational Kinematics

We define two reference frames as follows: $\hat{b} = [\hat{b}_1 \hat{b}_2 \hat{b}_3]^T$, an orthonormal set of unit vectors oriented along the space vehicle's principal axes; and $\hat{n} = [\hat{n}_1 \hat{n}_2 \hat{n}_3]^T$, an orthonormal set of unit vectors arbitrarily oriented, but fixed in inertial space. The relative orientation of the vehicle to the inertial axes is defined by

$$\{\underline{\hat{b}}\} = [C]\{\underline{\hat{n}}\} \tag{1}$$

where $[C]$ is the direction cosine matrix. In lieu of any three-angle description of orientation, we adopt the four Euler parameters defined as

$$\beta_0 = \cos(\phi/2) \qquad \beta_i = l_i \sin(\phi/2) \qquad (i = 1,2,3) \tag{2}$$

and

$$\underline{\beta} \triangleq [\beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3]^T$$

where

$$\underline{l} \equiv l_1 \hat{n}_1 + l_2 \hat{n}_2 + l_3 \hat{n}_3 \equiv l_1 \hat{b}_1 + l_2 \hat{b}_2 + l_3 \hat{b}_3$$

is the principal vector, and ϕ is the principal angle. Inspection of Eqs. (2) reveals the β's satisfy the constraint

$$\sum_{i=0}^{3} \beta^2 = 1 \tag{3}$$

The existence of $\hat{l}$ and ϕ, corresponding to arbitrary admissible values for the elements of $[C]$, is guaranteed by Euler's Principal Rotation Theorem[20-22] which states that a completely general reorientation of a rigid body can be accomplished by rigidly rotating through ϕ about $\hat{l}$. The

direction cosine matrix can be parameterized as a function of Euler parameters as

$$[C] = \begin{bmatrix} (\beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2) & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & (\beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2) & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & (\beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2) \end{bmatrix} \tag{4}$$

The Euler parameters can be related to any set of Euler angles by the method of Ref. 25. The Euler parameters' time derivatives are rigorously related to the angular velocity $\omega = (\omega_1\omega_2\omega_3)^T$ of $[\hat{b}]$ relative to $[\hat{n}]$ via the orthogonal kinematic relationship

$$\dot{\beta} = \underline{\underline{\Omega}}\beta = \underline{\underline{B}}\omega \tag{5}$$

where

$$\underline{\underline{\Omega}} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

$$\underline{\underline{B}} = \frac{1}{2} \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \quad \text{and} \quad (\cdot) = \frac{d}{dt}( )$$

Equation (5) presents a sharp contrast to the corresponding kinematical relationship for any three-angle description of orientation (which invariably contains ratios of transcendental functions of the angles, and has a geometric singularity in which two of the three angular rates tend to infinity for finite ω_i). Equation (5) has an implicit, exact integral $\Sigma_{i=0}^{i=3}\beta_i^2 = $ const. This constant should be unity, as is evident from Eq. (3), and is established by any valid choice of initial conditions.

B. Derivation of the Necessary Conditions from Pontryagin's Principle

To consider the rotational dynamics of a rigid space vehicle, we choose as state variables the four Euler parameters $(\beta_0, \beta_1, \beta_2, \beta_3)$ and the angular velocities $(\omega_1, \omega_2, \omega_3)$; the state differential equations are thus Eqs. (5) together with Euler's rotational equations of motion[23]

$$\dot{\underline{\omega}} = \underline{f}(\omega) + \underline{\underline{D}}u \tag{6}$$

where

$$\underline{f} = (-\vartheta_1\omega_2\omega_3, -\vartheta_2\omega_1\omega_3, -\vartheta_3\omega_1\omega_2)^T$$

$$\underline{\underline{D}} = \begin{bmatrix} I_1^{-1} & 0 & 0 \\ 0 & I_2^{-1} & 0 \\ 0 & 0 & I_3^{-1} \end{bmatrix}$$

where I_1, I_2 and I_3 are the spacecraft principal inertia, u_1, u_2, and u_3 are $\hat{b}_{,i}$ components of the control torque, and $\vartheta_1 \equiv (I_3 - I_2)/I_1$, $\vartheta_2 \equiv (I_1 - I_3)/I_2$, $\vartheta_3 = (I_2 - I_1)/I_3$. We seek a solution of Eqs. (5) and (6), satisfying the prescribed initial and final orientation and angular velocity given by

$$\beta_i(t_0) = \beta_{i0}, \omega_j(t_0) = \omega_{j0} \qquad \beta_i(t_f) = \beta_{if}, \omega_j(t_f) = \omega_{jf}$$

$$(i = 1,2,3,4; \quad j = 1,2,3) \tag{7}$$

Observe that the prescription of Euler parameter boundary conditions must be consistent with the constraint [see Eq. (3)] so that only 12 degrees of freedom exist.

We seek, in particular, the torque history $u_i(t)$ generating an optimal solution of Eqs. (5) and (6), satisfying the boundary conditions of Eq. (7), and which minimizes the performance index

$$J = \frac{1}{2} \int_{t_0}^{t_f} [u_1^2(\tau) + u_2^2(\tau) + u_3^2(\tau)] d\tau \tag{8}$$

We restrict attention to a piecewise continuous torque history $u_i(t)$.

The Hamiltonian function associated with minimizing Eq. (8) along trajectories of Eqs. (5) and (6) is

$$\mathcal{H} = \frac{1}{2}\underline{u}^T\underline{u} + \underline{\gamma}^T\underline{\underline{B}}\omega + \underline{\lambda}^T(\underline{f}(\underline{\omega}) + \underline{\underline{D}}u) \tag{9}$$

where λ_i and γ_j are co-state variables associated with β_i and ω_j. In addition to Eqs. (5) and (6), Pontryagin's Principle requires as necessary conditions that the λ's and γ's satisfy co-state differential equations derivable from

$$\dot{\underline{\gamma}} = -(\partial\mathcal{H}/\partial\underline{\beta})^T = \underline{\underline{\Omega}}\gamma \tag{10}$$

and

$$\dot{\underline{\lambda}} = -(\partial\mathcal{H}/\partial\underline{\omega})^T = -(\partial\underline{f}/\partial\underline{\omega})^T\underline{\lambda} - \underline{\underline{B}}^T\underline{\gamma}$$

where we have made use of the fact that $\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$, and $u_i(t)$ must be chosen at every instant so that the Hamiltonian $\mathcal{H}$ is minimized. This latter necessary condition requires (for $u_i(t)$ continuous) that

$$\partial\mathcal{H}/\partial u_i = 0 = u_i + \lambda_i/I_i$$

The optimal torque vector is determined as a function of the angular velocity co-state variables by minimization of the Hamiltonian to be

$$u_i = -\lambda_i/I_i \qquad (i = 1,2,3) \tag{11}$$

The state and co-state differential equations forming the boundary value problem are Eqs. (5) and (6) after using Eq. (11), together with the co-state equations.

State Equations

$$\dot{\underline{\beta}} = \underline{\underline{\Omega}}\beta \tag{12a}$$

$$\dot{\underline{\omega}} = \underline{f}(\omega) - \underline{\underline{D}}\underline{\underline{D}}\lambda \tag{12b}$$

Co-State Equations

$$\dot{\underline{\gamma}} = \underline{\underline{\Omega}}\gamma \tag{12c}$$

$$\dot{\underline{\lambda}} = -(\partial\underline{f}/\partial\underline{\omega})^T\underline{\lambda} - \underline{\underline{B}}^T\underline{\gamma} \tag{12d}$$

The next two sections deal with initializing and completion of a relaxation process for determination of the generally unknown $\gamma_i(t_0)$ and $\lambda_i(t_0)$ for the solution of the boundary-value problem. This relaxation process takes full account of

the fact that

$$\sum_{i=0}^{3} \gamma_i^2(t) = \text{const}$$

is a rigorous integral of Eq. (12c), but unlike Euler parameters, the constant cannot (in general) be taken as unity. This fact will be evident in the following developments.

III. Special Case Analytical Solutions

In general, Eqs. (12) do not admit analytical solutions, and numerical solutions must be iterated to acheive satisfaction of the terminal boundary conditions. However, certain boundary conditions zero certain terms of Eqs. (12) for all time, and thereby (without approximation) reduce them to specialized forms which can be solved analytically. Implicit special case solutions for the initial co-state boundary conditions are then achievable *without* iteration. These initial conditions will then be used to start the iterative relaxation process to solve the more general reorientation problem.

The three special case solutions correspond to "pure spin" reorientations about any one of the spacecraft's three principal axes of inertia. The corresponding boundary conditions for rotation about the *i*th axes are:

$$\beta_0(t_0) = \cos(\phi_0/2) \qquad \beta_j(t_0) = \sin(\phi_0/2)\delta_{ji}$$

$$\beta_0(t_f) = \cos(\phi_f/2) \qquad \beta_j(t_f) = \sin(\phi_f/2)\delta_{ji}$$

$$\omega_k(t_0) = \dot{\phi}_0 \delta_{ki} \qquad \omega_k(t_f) = \dot{\phi}_f \delta_{ki}$$

$$(j = 1,2,3; k = 1,2,3) \qquad (13)$$

where δ_{ij} is the Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

In all three cases, the initial angle ϕ_0, initial angular rate $\dot{\phi}_0$, final angle ϕ_f, and final angular rate $\dot{\phi}_f$ can be given arbitrary values. For example, we consider in detail the case $i = 1$. It can be seen by inspection that the 14 differential equations of Eq. (12) reduce to

$$\dot{\beta}_0 = -(\omega_1/2)\beta_1 \qquad \dot{\beta}_1 = (\omega_1/2)\beta_0 \qquad \dot{\beta}_2 = \dot{\beta}_3 = 0$$

$$\dot{\omega}_1 = -\lambda_1/I_1^2 \qquad \dot{\omega}_2 = \dot{\omega}_3 = 0$$

$$\dot{\gamma}_0 = -(\omega_1/2)\gamma_1 \qquad \dot{\gamma}_1 = (\omega_1/2)\gamma_0 \qquad \dot{\gamma}_2 = \dot{\gamma}_3 = 0$$

$$\dot{\lambda}_1 = \frac{1}{2}(\gamma_0\beta_1 - \gamma_1\beta_0) \qquad \dot{\lambda}_2 = \dot{\lambda}_3 = 0 \qquad (14)$$

which are shown in Ref. 25 to process the solutions

$$\beta_0(t) = \cos(\phi/2) \qquad \beta_1(t) = \sin(\phi/2) \qquad \beta_2(t) = \beta_3(t) = 0$$

$$\gamma_0(t) = -2I_1^2 \dddot{\phi}_0 \sin(\phi/2) \qquad \gamma_1(t) = 2I_1^2 \dddot{\phi}_0 \cos(\phi/2)$$

$$\gamma_2(t) = \gamma_3(t) = 0$$

$$\phi(t) = \phi_0 + \dot{\phi}_0(t - t_0) + \frac{1}{2}\ddot{\phi}_0(t - t_0)^2 + (1/6)\dddot{\phi}_0(t - t_0)$$

$$\omega_1(t) = \dot{\phi}(t) \qquad \omega_2(t) = \omega_3(t) = 0 \qquad (15)$$

$$\ddot{\phi}_0 = \frac{6(\phi_f - \phi_0)}{(t_f - t_0)^2} - \frac{2(2\dot{\phi}_0 + \dot{\phi}_f)}{(t_f - t_0)}$$

$$\dddot{\phi}_0 = -\frac{12(\phi_f - \phi_0)}{(t_f - t_0)^3} + \frac{6(\dot{\phi}_0 + \dot{\phi}_f)}{(t_f - t_0)^2}$$

$$\lambda_1(t) = -I_1^2\ddot{\phi}_0 - I_1^2 \dddot{\phi}_0(t - t_0) \qquad \lambda_2(t) = \lambda_3(t) = 0$$

$$u_1(t) = -\lambda_1(t)/I_1 = I_1[\ddot{\phi}_0 + \dddot{\phi}_0(t - t_0)] = I_1\dot{\omega}_1$$

$$u_2(t) = u_3(t) = 0$$

The key point of interest in the solutions just presented came in the evaluation of the free constants in the equation for λ_1. It was determined[25] that the norm $(\gamma^T(t)\gamma(t) = B = \text{const})$ for the angular velocity co-state is not unique, but is bounded below by a positive constant. This result motivated the formulation of the differential connection algorithm of Sec. IV based on the strategy that the magnitude of the angular velocity co-state be minimized.

The results just stated for $i = 1$ can be quickly paralleled to produce solutions for $i = 2,3$:

$$\gamma_0(t_0) = -2I_i^2 \dddot{\phi}_0\beta(t_0) \qquad \gamma_i(t_0) = -2I_i^2 \dddot{\phi}_0\beta_0(t_0)$$

$$\gamma_k(t_0) = 0 \qquad (k \neq i, \quad k = 1,2,3)$$

$$\lambda_i(t_0) = -I_i^2\ddot{\phi}_0 \qquad \lambda_k(t_0) = 0 \qquad (k \neq i; \quad k = 1,2,3)$$

$\phi_0, \dot{\phi}_0, \phi_f, \dot{\phi}_f$, given $\ddot{\phi}_0, \dddot{\phi}_0$ are found from Eqs. (15).

IV. A Relaxation Process for Solution of the Two-Point Boundary-Value Problem

We now consider a relaxation process for solution of the nonlinear two-point boundary-value problem; the initial step is to select, as a starting iterative, one of the above analytical solutions for pure spin about a principal axis (based, for example, upon which axis has the largest initial or final angular velocity component). The relaxation process occurs in the space of terminal boundary conditions and determines solutions to each of a sequence of boundary-value problems; the first being analytically solvable and the final being the actual desired boundary-value problem. The intermediate problems are merely an artifice to allow neighboring optimal trajectories' converged co-states to be employed as starting iteratives. By adaptively controlling the relaxation rate, we provide arbitrarily close starting iteratives for each subsequent iteration. This process represents a particular form of imbedding[13]; the desired extremal and an analytically solvable single axis extremal have been imbedded in a one parameter field of extremals.

One problem encountered when using the Euler parameter representation is that these parameters are redundant by virtue of the constraint [see Eq. (3)]. One must therefore be careful in constructing algorithms for solving the two-point boundary-value problem. The redundancy of the β's cited in Eq. (3) results in a corresponding redundancy in the co-state variables. Since the γ's satisfy a skew symmetric differential equation (12c) identical to the β's differential equation (12a), it is evident that

$$\gamma_0^2(t) + \gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) = \text{const} = B^2 \qquad (16)$$

However, as was discussed in Sec. III for the $i = 1$ solution, B is not unity, though bounded below. The fact that the initial γ's must, in general, be iteratively determined, motivates choosing some well-defined process for selecting a particular solution. "It seems altogether reasonable" to approach the general solution analogous to the approach of Ref. 25; we determine the initial co-state boundary conditions $\lambda_i(t_0)$ and $\gamma_i(t_0)$ which satisfy $\beta_i(t_f) = \beta_{if}$ and $\omega_i(t_f) = \omega_{if}$, with the criterion that

$$B^2 = \sum_{j=0}^{j=3} \gamma_j^2(t_0)$$

be minimized.

Since the terminal Euler parameters must satisfy Eq. (3), it is necessary to formally constrain only three of $\beta_i(t_f) = \beta_{if}$, where $i = 0, 1, 2, 3$; the remaining $\beta_i(t_f)$ will automatically satisfy Eq. (3) [by virtue of the fact that the constraint, cited in Eq. (3), is satisfied by any admissible initial state, and the skew symmetric differential equation (12a) has $\Sigma\beta_i^2 = \text{const}$ as a rigorous exact integral]. These considerations motivate a successive approximation strategy to solve the two-point boundary-value problem as follows:

Given starting estimates for the co-state vectors

$$\hat{\gamma}(t_0) = [\hat{\gamma}_0(t_0)\hat{\gamma}_1(t_0)\hat{\gamma}_2(t_0)\hat{\gamma}_3(t_0)]^T$$

$$\hat{\lambda}(t_0) = [\hat{\lambda}_1(t_0)\hat{\lambda}_2(t_0)\hat{\lambda}_3(t_0)]^T$$

we seek correction vectors $\Delta\gamma$ and $\Delta\lambda$ which minimize

$$B^2 = [\hat{\gamma}(t_0) + \Delta\gamma]^T [\hat{\gamma}(t_0) + \Delta\gamma]$$

$$= \hat{\gamma}_0(t_0)^T \hat{\gamma}(t_0) + 2\Delta\gamma^T \hat{\gamma}(t_0) + \Delta\gamma^T \Delta\gamma \qquad (17a)$$

subject to the terminal constraints

$$\beta_f' - \beta'[\hat{\gamma}(t_0) + \Delta\gamma, \hat{\lambda}(t_0) + \Delta\lambda, t_f] = 0$$

$$\omega_f - \omega[\hat{\gamma}(t_0) + \Delta\gamma, \hat{\lambda}(t_0) + \Delta\lambda, t_f] = 0 \qquad (17b)$$

where

$$\beta'(t) = [\beta_1(t)\beta_2(t)\beta_3(t)]^T \quad \omega(t) = [\omega_1(t)\omega_2(t)\omega_3(t)]^T$$

$$\beta_f' = \beta(t_f) \qquad \omega_f = \omega(t_f)$$

and the notations $\hat{\beta} = [a, b, t]$ and $\hat{\omega} = [a, b, t]$ denote the solutions of Eqs. (12) with initial co-states $\{\lambda(t_0) = b, \gamma(t_0) = a\}$. Assuming some process has led to specific fixed values for the initial β's and ω's, we proceed to develop a differential correction process for $\Delta\gamma$ and $\Delta\lambda$.

Linearizing Eqs. (17), we have

$$\beta_f - \hat{\beta} - A_{\beta\gamma}\Delta\gamma - A_{\beta\lambda}\Delta\lambda = 0 \qquad (18a)$$

$$\omega_f - \hat{\omega} - A_{\omega\gamma}\Delta\gamma - A_{\omega\lambda}\Delta\lambda = 0 \qquad (18b)$$

where $\hat{\beta}' \equiv \beta'[\hat{\gamma}(t_0), \hat{\lambda}(t_0), t_f]$ and $\hat{\omega} \equiv \omega[\hat{\gamma}(t_0), \hat{\lambda}(t_0)t_f]$ represent the solution of Eqs. (12) based upon current estimate $\hat{\gamma}(t_0)$ and $\hat{\lambda}(t_0)$ of the initial co-state.

$$A_{\beta\gamma} \equiv \left[\frac{\partial\beta[\hat{\gamma}(t_0),\lambda(t_0),t_f]}{\partial[\gamma(t_0)]} \bigg|_{\hat{\gamma}(t_0),\hat{\lambda}(t_0)} \right] = \left[\frac{\partial\beta_f}{\partial\gamma_0} \right] \qquad (19a)$$

$$A_{\beta\lambda} \equiv \left[\frac{\partial\beta'[\gamma(t_0),\lambda(t_0),t_f]}{\partial[\lambda(t_0)]} \bigg|_{\hat{\gamma}(t_0),\hat{\lambda}(t_0)} \right] = \left[\frac{\partial\beta_f'}{\partial\lambda_0} \right] \qquad (19b)$$

$$A_{\omega\gamma} \equiv \left[\frac{\partial\omega[\gamma(t_0),\lambda(t_0),t_f]}{\partial[\gamma(t_0)]} \bigg|_{\hat{\gamma}(t_0),\hat{\lambda}(t_0)} \right] = \left[\frac{\partial\omega_f}{\partial\gamma_0} \right] \qquad (19c)$$

$$A_{\omega\lambda} \equiv \left[\frac{\partial\omega[\gamma(t_0),\lambda(t_0),t_f]}{\partial[\lambda(t_0)]} \bigg|_{\hat{\gamma}(t_0),\hat{\lambda}(t_0)} \right] = \left[\frac{\partial\omega_f}{\partial\lambda_0} \right] \qquad (19d)$$

The calculation of these derivatives is a separate issue, dealt with in Appendix 1 of Ref. 25.

Observe that Eq. (17a), to be minimized, does not explicitly depend upon $\Delta\lambda$, but $\Delta\gamma$ and $\Delta\lambda$ are constrained by Eqs. (18); thus $\Delta\lambda$ cannot be assigned arbitrary values. Since $A_{\omega\lambda}$ is square and presumed to be nonsingular, we can determine $\Delta\lambda$ from Eq. (18b) as a function of $\Delta\gamma$ by

$$\Delta\lambda = A_{\omega\gamma}^{-1}(\omega_f - \hat{\omega}) - A_{\omega\lambda}^{-1}A_{\omega\gamma}\Delta\gamma \qquad (20)$$

Substitution of Eq. (20) into Eq. (18a) then replaces Eqs. (18) by a single constraining relationship depending only upon $\Delta\gamma$:

$$(\beta_r - \hat{\beta}) - A_{\beta\lambda}A_{\omega\lambda}^{-1}(\omega_f - \hat{\omega}) - \tilde{A}\Delta\gamma = 0 \qquad (21)$$

where

$$\tilde{A} \equiv A_{\beta\gamma} - A_{\beta\lambda}A_{\omega\lambda}^{-1}A_{\omega\gamma} \qquad (22)$$

Using the Lagrange multiplier rule to minimize Eq. (17a) subject to the constraint cited in Eq. (21), we introduce the augmented function

$$\Phi = \hat{\gamma}^T(t_0)\hat{\gamma}(t_0) + 2\hat{\gamma}^T(t_0)\Delta\gamma + \Delta\gamma^T\Delta\gamma$$

$$+ \underline{\Lambda}^T[(\beta_f - \hat{\beta}) - A_{\beta\lambda}A_{\omega\lambda}^{-1}(\omega_f - \hat{\omega}) - \tilde{A}\Delta\gamma] \qquad (23)$$

where Λ is a 3×1 vector of Lagrange multipliers. We seek corrections $\Delta\gamma$ to the initial co-state which minimize Eq. (23); as a necessary condition, we require

$$\frac{\partial\Phi}{\partial(\Delta\gamma)} = 0 = 2\hat{\gamma}(t_0) + 2\Delta\gamma - \tilde{A}^T\underline{\Lambda} \qquad (24)$$

Since the function [see Eqs. (17)] is a positive definite quadratic form, and the constraint cited in Eq. (21) is linear, it follows that sufficient conditions are satisfied, and Φ is uniquely minimized by the stationary point satisfying Eq. (24). The optimum $\gamma(t_0)$ corrections, in terms of $\underline{\Lambda}$, follow from Eq. (24) as

$$\Delta\gamma = \frac{1}{2}\tilde{A}^T\underline{\Lambda} - \hat{\gamma}(t_0) \qquad (25)$$

Substitution of Eq. (25) into the constraint cited in Eq. (21) yields a solution for the multipliers as

$$\frac{1}{2}\underline{\Lambda} = (\tilde{A}\tilde{A}^T)^{-1}[(\beta_f - \hat{\beta}) - A_{\beta\lambda}A_{\omega\lambda}^{-1}(\omega_f - \hat{\omega}) + \tilde{A}\hat{\gamma}(t_0)] \qquad (26)$$

Substitution of Eq. (26) into Eq. (25) yields the solution for $\Delta\gamma$ as

$$\Delta\gamma = \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}[(\beta_f - \beta') - A_{\beta\lambda}A_{\omega\lambda}^{-1}(\omega_f - \hat{\omega})$$

$$+ \tilde{A}\hat{\gamma}(t_0)] - \hat{\gamma}(t_0) \qquad (27)$$

The solution for $\Delta\lambda$ then follows immediately from Eq. (20).

This discussion can all be summarized as the differential correction algorithm, diagrammed in Fig. 1, for refining given approximate intial co-states $\gamma(t_0)$ and $\lambda(t_0)$ so that a solution is achieved (provided the starting estimates are "sufficiently good").

Let us consider a relaxation process which should effectively guarantee that the algorithm diagrammed in Fig. 1 will work reliably. The only significant assumption en route to the algorithm was the local linearization of Eqs. (17) to obtain Eqs. (18). Ignoring certain singular events‡ [leading to the inverses in Eqs. (20) and (27) not existing], we can expect this algorithm to converge if the starting estimates $\hat{\gamma}(t_0)$ and $\hat{\lambda}(t_0)$ are sufficiently close to their true values. We describe an "adaptive relaxation process" in which we can always obtain starting estimates with arbitrarily small displacements from their true values.

Use is made of the fact that the necessary conditions can be rigorously satisfied if the boundary conditions belong to either of the three sets defined by Eq. (13). Defining the

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‡These singular events are not believed to be of practical concern; the matrices are of low order, and have been observed to be well-conditioned.
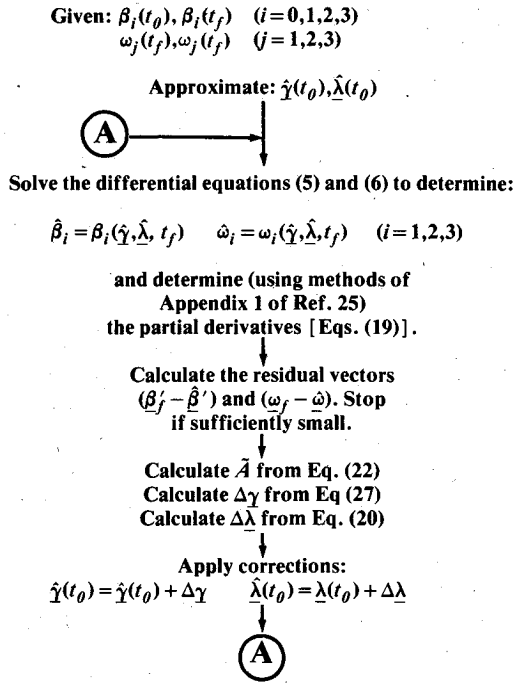
Given: $\beta_i(t_0), \beta_i(t_f)$ $(i=0,1,2,3)$
$\omega_j(t_f), \omega_j(t_f)$ $(j=1,2,3)$

Approximate: $\hat{\gamma}(t_0), \underline{\hat{\lambda}}(t_0)$

(A) ───────▶|

Solve the differential equations (5) and (6) to determine:

$\hat{\beta}_i = \beta_i(\hat{\underline{\gamma}}, \underline{\hat{\lambda}}, t_f)$ $\hat{\omega}_i = \omega_i(\hat{\underline{\gamma}}, \underline{\hat{\lambda}}, t_f)$ $(i=1,2,3)$

and determine (using methods of
Appendix 1 of Ref. 25)
the partial derivatives [Eqs. (19)].

Calculate the residual vectors
$(\underline{\beta}_f - \hat{\underline{\beta}}')$ and $(\underline{\omega}_f - \hat{\underline{\omega}})$. Stop
if sufficiently small.

Calculate $\bar{A}$ from Eq. (22)
Calculate $\Delta\underline{\gamma}$ from Eq (27)
Calculate $\Delta\underline{\lambda}$ from Eq. (20)

Apply corrections:
$\hat{\underline{\gamma}}(t_0) = \hat{\underline{\gamma}}(t_0) + \Delta\underline{\gamma}$ $\underline{\hat{\lambda}}(t_0) = \underline{\lambda}(t_0) + \Delta\underline{\lambda}$

(A)

Fig. 1 **Differential correction algorithm for determination of initial co-state variables.**

sequence of boundary conditions

$$\underline{X}_n = [\theta_{1n}(t_0)\theta_{2n}(t_0)\theta_{3n}(t_0) \vdots \omega_{1n}(t_0)\omega_{2n}(t_0)\omega_{3n}(t_0) \vdots$$

$$\theta_{1n}(t_f)\theta_{2n}(t_f)\theta_{3n}(t_f) \dot{\omega}_{1n}(t_f) \omega_{2n}(t_f)\omega_{3n}(t_f)]^T$$

$$(n=0,1,...,N) \qquad (28)$$

$\underline{X}_N = \underline{X}_{\text{true}}$ = the true desired boundary conditions

$\underline{X}_0 = \underline{X}_{\text{start}}$ = a set of boundary conditions for which the initial co-state variables can be determined exactly without iteration [i.e., a set of boundary conditions belonging to the sets defined by Eq. (13)].

where $\beta_{0n}(t_0)$ and $\beta_{fn}(t_f)$ are determined in terms of the sequence of 1-2-3 Euler angles θ_{1n}, θ_{2n} and θ_{3n} by the transformation found in Ref. 25. Let

$$\underline{P}_n = [\gamma_{0n}(t_0)\gamma_{1n}(t_0)\gamma_{2n}(t_0)\gamma_{3n}(t_0) \vdots \lambda_{1n}(t_0)\lambda_{2n}(t_0)\lambda_{3n}(t_0)]^T$$

$$= [\underline{\gamma}_n^T(t_0) \vdots \underline{\lambda}_n^T(t_0)]^T \quad (n=0,1,2,...,N) \qquad (29)$$

be the sequence of co-states corresponding to the solution of Eqs. (5) and (6) connecting the upper and lower partition of $\underline{X}_n$. Initially, only the co-state $\underline{P}_0$ is known.

We introduce the relaxation parameter α such that

$$0 \le \alpha_n \le 1 \quad \{0 \le \alpha_1 < \alpha_2 < \alpha_3... < \alpha_N = 1\}$$

and set the boundary conditions for the nth boundary-value problem as

$$\underline{X}_n = \underline{X}_0 + \alpha_n [\underline{X}_{\text{true}} - \underline{X}_{\text{start}}] \qquad (30)$$

For example, set

$$\alpha_n = n/N \qquad (31)$$

then, clearly Eq. (30) satisfies the desired conditions

$$\underline{X}_0 = \underline{X}_{\text{start}} \quad \text{and} \quad \underline{X}_N = \underline{X}_{\text{true}} \qquad (32)$$

More generally, one could adjust α_n adaptively, based upon the efficiency of convergence at step n-1. In all cases, we require that Eq. (32) be satisfied. In the numerical example, we provide results which simply use Eq. (31); modifying the algorithm to make α control adaptive has not been found necessary, but this may be due to the limited variety of maneuvers studied to date.

The starting estimate for the co-state vector [see Eq. (29)] for the nth step in the relaxation process is obtained from

$$\tilde{\underline{P}}_n = \underline{P}_{n-1} + (\alpha_n - \alpha_{n-1}) \left\{ \frac{d\underline{P}}{d\alpha} \bigg|_{n-1} \right\} \quad (n>1) \qquad (33)$$

where the derivative of the co-state vector with respect to the relaxation parameter α is approximated via finite differences as

$$\left\{ \frac{d\underline{P}}{d\alpha} \bigg|_{n-1} \right\} = \begin{cases} (\alpha_{n-1} - \alpha_{n-2})^{-1} \{\underline{P}_{n-1} - \underline{P}_{n-2}\} & (n>1) \\ \{0\} & (n=1) \end{cases} \qquad (34)$$

and where $\underline{P}_n$ are the converged co-state vectors [see Eq. (29)] resulting from previous applications of the algorithm of Fig. 1. An accelerated quadratic estimate for the initial co-state vector [see Eq. (34)] is obtained from

$$\hat{\underline{P}}_n = \tilde{\underline{P}}_n + \underline{\Delta P}_{n-1} + (\alpha_n - \alpha_{n-1}) \left\{ \frac{d(\underline{\Delta P}_n)}{d\alpha} \bigg|_{n-1} \right\} \quad (n>2) \qquad (35)$$

where $\underline{\Delta P}_n$ represents the actual error in the linearly predicted co-states in the nth relaxation state. The derivative of the linear co-state prediction error with respect to the relaxation parameter α is approximated via finite differences as

$$\left\{ \frac{d(\underline{\Delta P}_n)}{d\alpha} \bigg|_{n-1} \right\} = \begin{cases} (\alpha_{n-1} - \alpha_{n-2})^{-1} \{\underline{\Delta P}_{n-1} - \underline{\Delta P}_{n-2}\} & (n>3) \\ \{0\} & (n \le 2) \end{cases} \qquad (36)$$

Each increment (controlled via specification of α_n) of the state boundary conditions [see Eq. (30)] is thus supported by a linear or better approximation of the co-state boundary conditions via Eq. (35), followed by a differential correction refinement using the algorithm of Fig. 1 (to isolate the converged co-state $\underline{P}_n$ to desired accuracy). Observe that the process is initiated at a converged $\underline{P}_0$ vector, and if convergence difficulties are encountered at any step of the process, one simply reduces the value of $(\alpha_n - \alpha_{n-1})$ [thereby making the error in the extrapolations, see Eqs. (33) or (35), arbitrarily small]. The numerical results summarized in Sec. V support the conclusion that this relaxation process is not only reliable, but has also been found to be reasonably efficient.

V. Numerical Tests

We summarize two numerical examples to illustrate the developments previously stated.

Case I is near-trivial as a numerical example, but it serves as an important validation role (since it can be rigorously solved without iteration). Case I is a "rest-to-rest" maneuver corresponding to a 90 deg reorientation about the $\hat{b}_1$ principal axis, with zero initial and final angular velocity. Table 1 summarizes the initial and final boundary conditions. Recognizing the case I boundary conditions as belonging to the set 1 family, we calculate the following initial co-state:

$$\underline{\gamma}^T(t_0) = \{0, -37.699118, 0, 0\}$$

$$\underline{\lambda}^T(t_0) = \{-9.424778, 0, 0\} \qquad (37)$$

Using this initial co-state and the case I (Table 1) initial state, Eqs. (12) were integrated numerically (via a four-cycle Runge-Kutta algorithm at a constant step size of 0.01s.) The numerical solution agreed at every step with the analytical
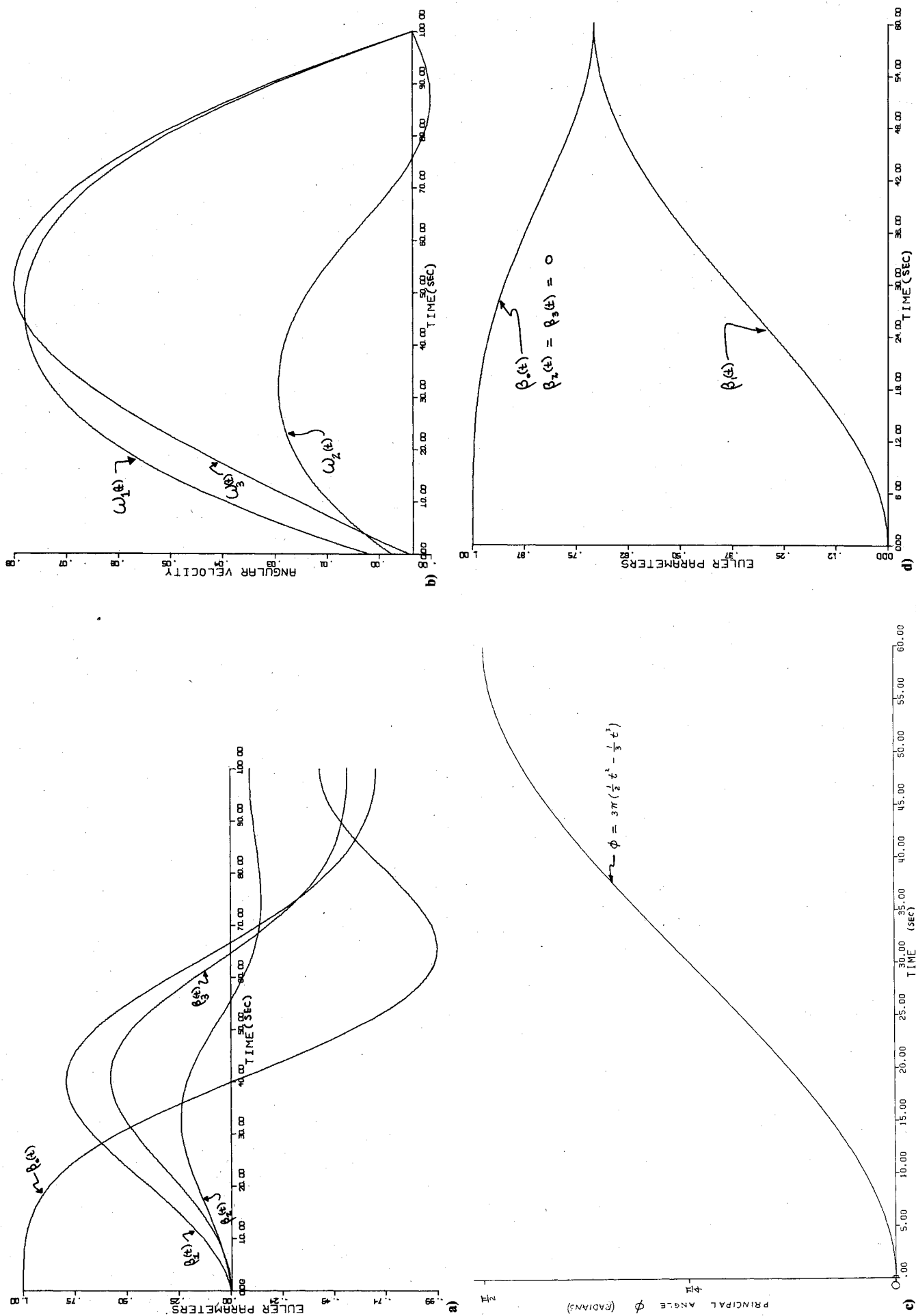
Fig. 2 Case I optimal maneuver.

Table 1　Assumed data for cases I and II

	Case I maneuver $(I_1=1, I_2=0.8, I_3=0.5)$		Case II maneuver $(I_1=1\times10^6$ kg·m^2, $I_2=833333$ kg·m^2, $I_3=916667$ kg·m^2)	
	Initial ($t_0=0$) state	Final ($t_f=1$ min) state	Initial ($t_0=0$) state	Final ($t_f=100$ s) state
θ_1	0	$\pi/2$	0	$\pi/2$
θ_2	0	0	0	$\pi/3$
θ_3	0	0	0	$\pi/4$
β_0	1	0.70711	1	-0.43047
β_1	0	0.70711	0	-0.70106
β_2	0	0	0	-0.09230
β_3	0	0	0	-0.56098
ω_1	0	0	0.01 rad/s	0
ω_2	0	0	0.005	0
ω_3	0	0	0.001	0

Table 2　Boundary condition relaxation for case III

	$\alpha_n \rightarrow$	0	0.001	0.25	0.50	0.75	1.00
	Tolerance$^a \rightarrow$	...	10^{-2}	10^{-2}	10^{-2}	10^{-2}	0^{-5}
	No. of iterations →	...	1	4	8	6	4
Initial state	$\theta_1(t_0)$	0	0	0	0	0	0
	$\theta_2(t_0)$	0	0	0	0	0	0
	$\theta_3(t_0)$	0	0	0	0	0	0
	$\omega_1(t_0)$	0.01 rad/s	0.0100	0.0100	0.0100	0.0100	0.01
	$\omega_2(t_0)$	0	0.00005	0.00125	0.00250	0.00375	0.005
	$\omega_3(t_0)$	0	0.00001	0.00025	0.00050	0.00075	0.001
Final state	$\theta_1(t_f)$	$5\pi/2$	$5\pi/2$	$5\pi/2$	$5\pi/2$	$5\pi/2$	7.85397 ($\equiv5\pi/2$)
	$\theta_2(t_f)$	0	0.01048	0.260829	0.52249	0.78471	1.04731 ($\equiv\pi/3$)
	$\theta_3(t_f)$	0	0.07857	0.19620	0.39248	0.58909	0.78546 ($\equiv\pi/4$)
	$\omega_1(t_f)$	0	0	0	0	0	0
	$\omega_2(t_f)$	0	0	0	0	0	0
	$\omega_3(t_f)$	0	0	0	0	0	0
Converged co-state ($\times10^9$)	$\gamma_0(t_0)$	-4.3124	-4.31237	-4.22029	-4.01835	-3.81454	-3.64945
	$\gamma_1(t_0)$	0	-0.01576	-0.37601	-0.67991	-0.91423	-1.10589
	$\gamma_2(t_0)$	0	-0.03034	-0.73721	-1.34988	-1.80119	-2.13870
	$\gamma_3(t_0)$	0	0	0	0	0	0
	$\lambda_1(t_0)$	-0.17649	-0.17649	-0.17685	-0.17546	-0.17082	-0.16634
	$\lambda_2(t_0)$	0	-0.00063	-0.01501	-0.02674	-0.03496	-0.04117
	$\lambda_3(t_0)$	0	-0.00039	-0.01101	-0.02672	-0.04411	-0.05838

^{a}Maximum relative error in ω's (normalized by 0.001), and β's (normalized by 1).

solution [see Eqs. (15)] to seven digits. This test provides confidence that the general software has been correctly coded. The control and state variables along this maneuver are sketched in Figs. 2a-2d.

The Case II boundary conditions (Table 1) define a rather general "de-tumble" maneuver which is not tractable (as a closed-form analytical solution). Since the vehicle is initially most nearly rotating about the $\hat{b}_1$ axis (as is evidenced by the relative magnitude of the initial ω_i's), the decision is made to

use set 1 boundary conditions to start the relaxation process of Sec. IV. Table 2 lists the sequence of boundary-value settings and converged co-states resulting from the relaxation process. In this particular case, we took $N=5$ in Eq. (31); we found most reliable and efficient convergence ensued using the differential correction algorithm of Fig. 1. For each of the five relaxation stages, an average of four differential corrections were required. The rigorous partial derivatives of the state transition matrix were computed and used for the
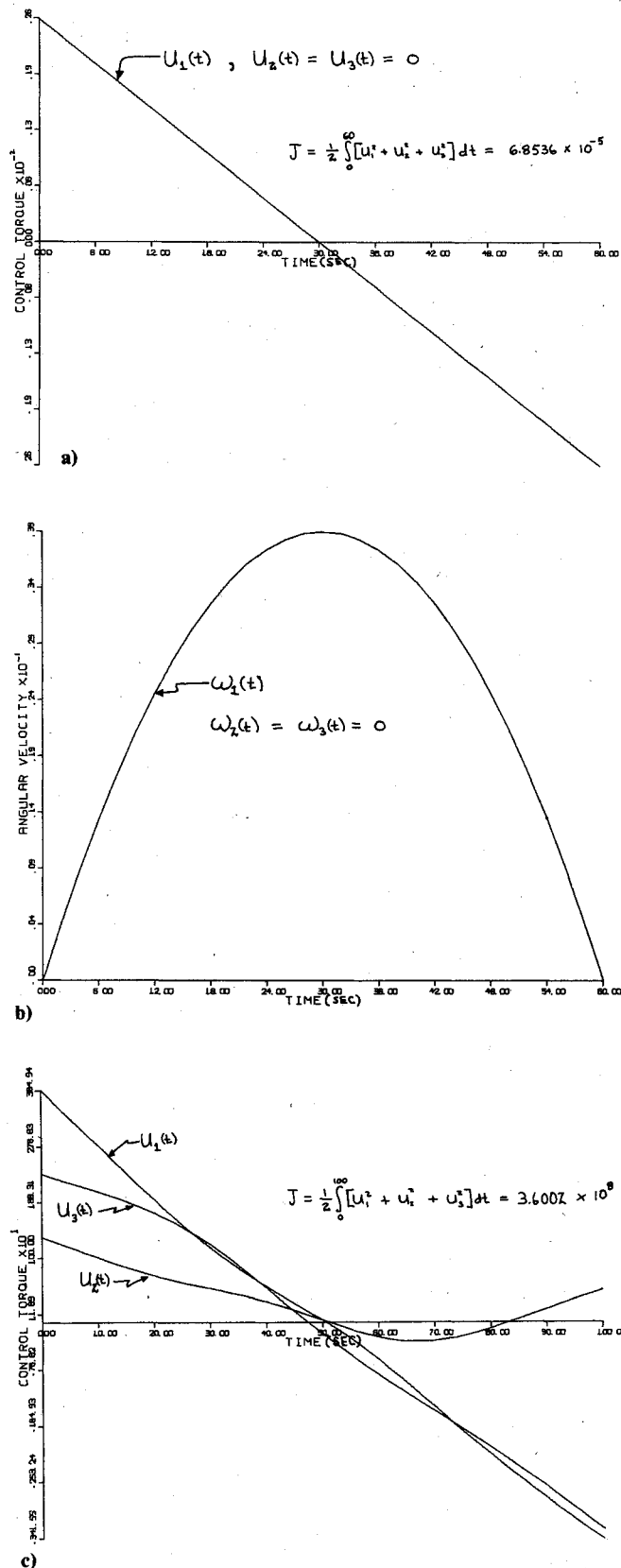
$$U_1(t) \quad , \quad U_2(t) = U_3(t) = 0$$

$$J = \tfrac{1}{2}\int_0^{60}[u_1^2 + u_2^2 + u_3^2]\,dt = 6.8536 \times 10^{-5}$$

a)

$$\omega_1(t)$$

$$\omega_2(t) = \omega_3(t) = 0$$

b)

$$U_1(t)$$

$$U_3(t)$$

$$U_2(t)$$

$$J = \tfrac{1}{2}\int_0^{100}[u_1^2 + u_2^2 + u_3^2]\,dt = 3.6007 \times 10^8$$

c)

Fig. 3   Case II optimal maneuver.

first differential correction for each relaxation state, with approximate partial derivatives being generated for successive differential corrections by the method of Ref. 18. The final optimal maneuver, which converged to seven digits, is sketched in Figs. 3a-3c. This example appears to be of representative difficulty; the unqualified successful determination of this optimal maneuver (and several parameter

variations thereof) provides the basis for our guarded optimism. We believe that this approach will work in general.

VI. Concluding Remarks

The results of the present paper provide a basis for a systematic solution for optimal spacecraft rotational maneuvers under the assumption of an asymmetric rigid body model. The relaxation method has worked exceptionally well on the numerical examples studied.

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